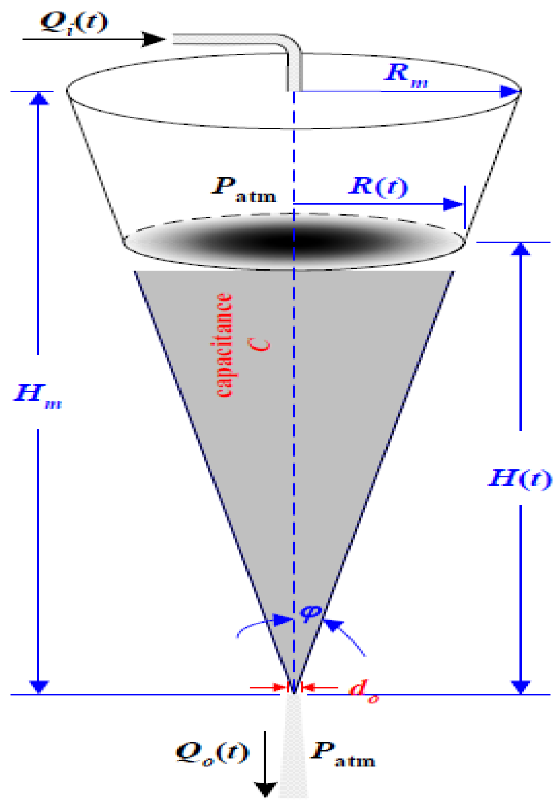
# CITY COLLEGE

# CITY UNIVERSITY OF NEW YORK




## HOMEWORK #6

# NONLINEAR ANALYSIS OF LIQUID DRAINAGE FROM A CONICAL TANK IN REVERSE POSITION

**ME 411: System Modeling Analysis and Control**

**Fall 2010**

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## 1.0 <u>**Transfer functions & governing equations**</u>.

<u>Governing Equations:</u>

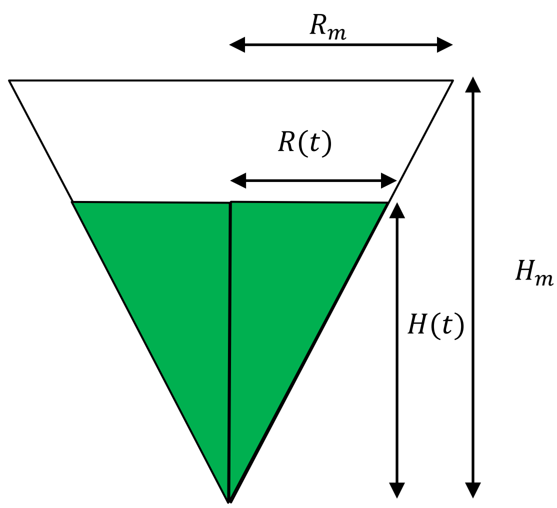
Since the volume of the liquid in the conical tank is:

$$V = \frac{1}{3}\pi R^2 H ------(1)$$

And its compliance is given by

$$C(H) = \frac{dV(H)}{dH} ------------- (2)$$

From the geometry of the reverse cone,




$$R(t) = \frac{R_m}{H_m} H(t) -----------(3)$$

From Equations (1), (2) and (3)

$$C(H) = \pi \left(\frac{R_m}{H_m}\right)^2 H^2 ---------(4)$$

From the laws of conservation of mass, we have:

$$\rho C(H) dH = \rho[Q_i(t) - Q_0(t)]dt$$

$$\cancel{\rho}\pi \left(\frac{R_m}{H_m}\right)^2 H(t)^2 dH = \cancel{\rho}[Q_i(t) - Q_0(t)]dt$$

We have

$$Q_0(t) = C_d A_o \sqrt{\frac{2\Delta P(t)}{\rho}} \text{ where}$$

$$\Delta P(t) = P_{before\ orifice} - P_{afterorifice} = \rho g H(t)$$

Hence

$$\pi \left(\frac{R_m}{H_m}\right)^2 H(t)^2 dH = \left[Q_i(t) - C_d A_o \sqrt{\frac{2\rho g H(t)}{\rho}}\right] dt$$

$$\pi \left(\frac{R_m}{H_m}\right)^2 H(t)^2 \frac{dH}{dt} = \left[Q_i(t) - C_d A_o \sqrt{\frac{2\cancel{\rho} g H(t)}{\cancel{\rho}}}\right]$$

$$\pi \left(\frac{R_m}{H_m}\right)^2 H(t)^2 \frac{dH}{dt} + C_d A_o \sqrt{2g}\, H(t)^{0.5} = Q_i(t)$$

This is ***first order nonlinear differential equations***.

$$\pi \left(\frac{R_m}{H_m}\right)^2 H(t)^2 \frac{dH}{dt} + C_d A_o \sqrt{2g}\, H(t)^{0.5} = Q_i(t)$$

$$\frac{\pi \left(\frac{R_m}{H_m}\right)^2 H(t)^2}{\left(Q_i(t) - C_d A_o \sqrt{2g}\, H(t)^{0.5}\right)} dH = dt$$

$$\int_0^t dt = \int_{H_i}^{H(t)} \frac{\pi \left(\frac{R_m}{H_m}\right)^2 H(t)^2}{\left(Q_i(t) - C_d A_o \sqrt{2g}\, H(t)^{0.5}\right)} dH$$

Where,

$R_m = Radius\ of\ the\ cone\ at\ the\ base, m$

$H_m = total\ height\ of\ the\ cone, m$

$C_d = cofficient\ of\ drag\ at\ orifice$

$A_0 = orifice\ area, m^2$

$Q_i(t) = rate\ of\ inital\ discharge, \frac{L}{s}$

$Q_0(t) = rate\ of\ \ discharge\ at\ orifice, \frac{L}{s}$

$\rho = density\ of\ fluid, \frac{kg}{m^3}$

$g = gravity, \frac{m}{s^2}$

## 2.0 Analytical Solution of governing equations:

$$\int_0^t dt = \int_{H_,}^{H(t)} \frac{\pi \left(\frac{R_m}{H_m}\right)^2 H(t)^2}{\left(Q_i(t) - C_d A_o \sqrt{2g}\, \sqrt{H(t)}\right)} dH$$

Assume

$$\pi \left(\frac{R_m}{H_m}\right)^2 = X\ and C_d A_o \sqrt{2g} = B\ and\ \sqrt{H(t)} = u$$

Then,

$$\frac{du}{dH} = \frac{1}{2\sqrt{H(t)}} = \frac{1}{2u} \rightarrow dH = 2udu$$

$$t = \int_{u_i}^{u(t)} \frac{X * u^4}{(\bar{Q_\iota} - Y\, u)} 2udu$$

$$t = \int_{u_i}^{u(t)} \frac{2 * X * u^5}{(\bar{Q_\iota} - Y\, u)} du$$

$$t = -\left(\frac{2}{5}\right) * X * \frac{u^5}{B} - \left(\frac{1}{2}\right) * X * Q * \frac{u^4}{B^2} - \left(\frac{2}{3}\right) * X * Q^2 * \frac{u^3}{B^3} - X * Q^3 * \frac{u^2}{B^4} - 2 * X * Q^4 * \frac{u}{B^5} - 2 * X * Q^5 * \frac{\ln(-Q + B * u)}{B^6} \Bigg|_{u_i}^{u(t)}$$

$$t = -\left(\frac{2}{5}\right) * X * \frac{H^{\frac{5}{2}}}{B} - \left(\frac{1}{2}\right) * X * Q * \frac{H^2}{B^2} - \left(\frac{2}{3}\right) * X * Q^2 * \frac{H^{\frac{3}{2}}}{B^3} - X * Q^3 * \frac{H}{B^4} - 2 * X * Q^4 * \frac{\sqrt{H}}{B^5} - 2 * X * Q^5 * \frac{ln\left(-Q + B * \sqrt{H}\right)}{B^6} \Bigg|_{H_i}^{H(t)}$$

$$t = -\left(\frac{2}{5}\right) * X * \frac{H(t)^{\frac{5}{2}}}{B} - \left(\frac{1}{2}\right) * X * Q * \frac{H(t)^2}{B^2} - \left(\frac{2}{3}\right) * X * Q^2 * \frac{H(t)^{\frac{3}{2}}}{B^3} - X * Q^3 * \frac{H(t)}{B^4} - 2 * X * Q^4 * \frac{\sqrt{H(t)}}{B^5} - 2 * X * Q^5 * \frac{ln\left(-Q + B * \sqrt{H(t)}\right)}{B^6} + \left(\frac{2}{5}\right) * X * \frac{H_i^{\frac{5}{2}}}{B} + \left(\frac{1}{2}\right) * X * Q * \frac{H_i^2}{B^2} + \left(\frac{2}{3}\right) * X * Q^2 * \frac{H_i^{\frac{3}{2}}}{B^3} + X * Q^3 * \frac{H_i}{B^4} + 2 * X * Q^4 * \frac{\sqrt{H_i}}{B^5} + 2 * X * Q^5 * \frac{ln\left(-Q + B * \sqrt{H_i}\right)}{B^6}$$

$$t = \left(\frac{2}{5}\right) * \frac{X}{B} * \left(H_i^{\frac{5}{2}} - H(t)^{\frac{5}{2}}\right) + \left(\frac{1}{2}\right) * \frac{XQ}{B^2} * \left(H_i^2 - H(t)^2\right) + \left(\frac{2}{3}\right) * \frac{XQ^2}{B^3} * \left(H_i^{\frac{3}{2}} - H(t)^{\frac{3}{2}}\right) - \frac{XQ^3}{B^4} * \left(H_i - H(t)\right) - 2 * \frac{XQ^4}{B^5} * \left(\sqrt{H_i} - \sqrt{H(t)}\right) + 2 * \frac{XQ^5}{B^6} * ln\left(\frac{Q - B * \sqrt{H_i}}{Q - B * \sqrt{H(t)}}\right)$$

## 3.0

| $C_d$ | $R_m$ $[m]$ | $H_m$ $[m]$ | $d_0$ $[m]$ | $H_0$ $[m]$ | $\bar{Q}_i$ $\left[\frac{L}{s}\right]$ |
|---|---|---|---|---|---|
| **0.6** | **1** | **2** | **0.05** | **2** | **4** |

With the data together, Using the trapezoidal rule for numerical integration to find the liquid height H (t) and the outflow rate $Q_0$ (t). Plot H (t) and $Q_0$ (t) vs. time (t) for two case.

$$\int_0^t dt = \int_{H_,}^{H(t)} \frac{\pi \left(\frac{R_m}{H_m}\right)^2 H(t)^2}{\left(Q_i(t) - C_d A_o \sqrt{2g}\, H(t)^{0.5}\right)} dH$$

# Methods:

## Trapezoidal Rule

### Free Response

Estimate the settling time required to drain out the tank (say, within 1% of the initial height). $H(t)|_{t=0} = H_0$ and $Q(t)|_{t=0} = 0$.

$$t_n = t_{n-1} - \left[ \frac{\pi \left(\frac{R_m}{H_m}\right)^2 H_n}{C_d A_o \sqrt{2g}\, {H_n}^{0.5}} + \frac{\pi \left(\frac{R_m}{H_m}\right)^2 H_{n-1}}{C_d A_o \sqrt{2g}\, {H_{n-1}}^{0.5}} \right] * \frac{H_n - H_{n-1}}{2}$$

### Step Response

$$\mathrm{Q_i(t)} = \mathrm{Q_i(t)} * 1(\mathrm{t}) = \begin{Bmatrix} \overline{\mathrm{Q_i}} \text{ for } t > 0 \\ 0 \text{ for } t < 0 \end{Bmatrix} \text{ and H (t) for t=0 equals 0.}$$

Estimate the 99% steady state liquid height, the 99% steady state outflow rate and 99% settling time from the plot. Compare the numerical results with theoretical predictions. $H_{s.s} = \lim_{t\to\infty} H(t)$ and $Q_{os.s} = \lim_{t\to\infty} Q_o(t)$.

$$t_n = t_{n-1} + \left[ \frac{\pi \left(\frac{R_m}{H_m}\right)^2 H_n}{\overline{\mathrm{Q_i}} - C_d A_o \sqrt{2g}\, {H_n}^{0.5}} + \frac{\pi \left(\frac{R_m}{H_m}\right)^2 H_{n-1}}{\overline{\mathrm{Q_i}} - C_d A_o \sqrt{2g}\, {H_{n-1}}^{0.5}} \right] * \frac{H_n - H_{n-1}}{2}$$

# Euler method:

Approximating the differential equations differential equations operator $\frac{d}{dt}$ with the difference operator $\frac{\Delta}{\Delta t}$, the 1st order nonlinear differential equations becomes:

$$\pi \left(\frac{R_m}{H_m}\right)^2 H(t)^2 \frac{\Delta H(t)}{\Delta t} + C_d A_o \sqrt{2g}\, H(t)^{0.5} = Q_i(t)$$

$$\pi \left(\frac{R_m}{H_m}\right)^2 H(t)^2 \frac{H_n - H_{n-1}}{\Delta t} + C_d A_o \sqrt{2g}\, (H_{n-1})^{0.5} = Q_{i_{n-1}}$$

$$H_n = \frac{Q_{i_{n-1}} - C_d A_o \sqrt{2g}\, (H_{n-1})^{0.5}}{\pi \left(\frac{R_m}{H_m}\right)^2 (H_{n-1})^2} \Delta t + H_{n-1}$$

# 4.0 MatLab code:

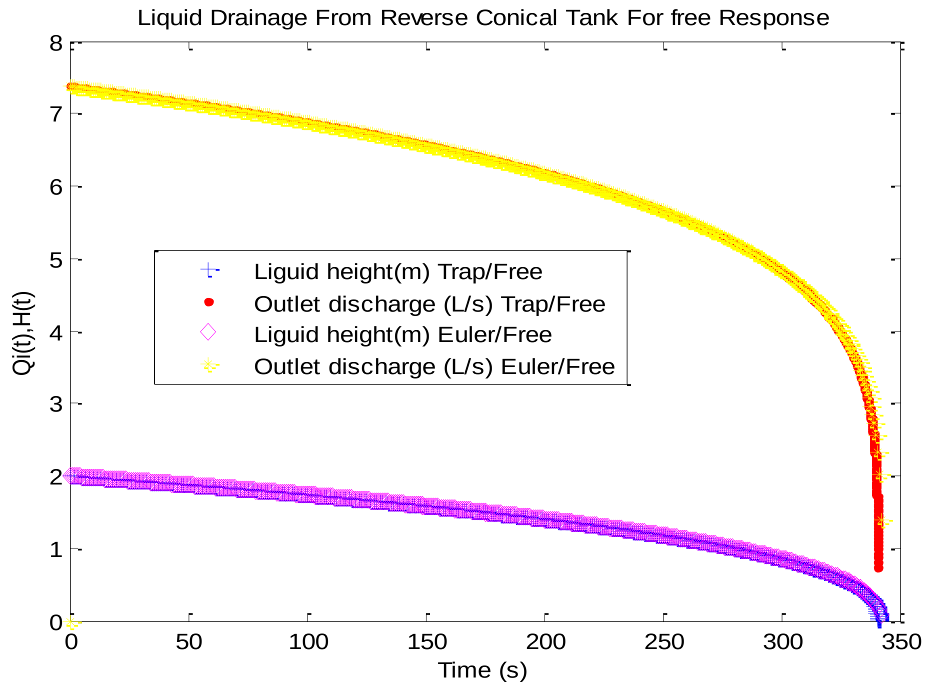
```
% FREE RESPONSE ANALYSIS FOR INVERSE CONE
clear all
clc
% Intial Inputs:
C_d=0.6;     %Orifice draf cofficient
R_m=1;       %Radius of base of cone,m
H_m=2;       %Total height of cone,m
g=9.81;      %Acceleration due to gravity,m
d_o=0.05;    %Diameter of outlet,m


%Calculation
A_o=pi*((d_o)^2)/4; %cross sectional area of outlet, m^2

% Free Response Trapozoidal Method
h=2;                    %Initial height of liquid level,m
hf=0.01*h;              %Final height of fluid in cone,m
ite=501;                %Number of steps
A=pi*(R_m/H_m)^2;       %Coffecient for fluid capacitance
B=C_d*A_o*sqrt(2*g);    %Coeffecient for outflow, m^(3/2)/s
h_step=(h-hf)/(ite-1);  %Step size of height,m

for n=1:ite;
    hf_F(n)=h-(n-1)*h_step;
    Q_o(n)=B*sqrt(hf_F(n));
    Q_oL(n)=Q_o(n)*1000;
    Q_i(n)=0;
    C_F(n)=pi*(R_m/H_m)^2*hf_F(n)^2;
end;

t_F(1)=0;
h_F(1)=0;
for n=2:ite
    t_F(n)=t_F(n-1)+(C_F(n-1)/(Q_i(n-1)-Q_o(n-1))+C_F(n)/(Q_i(n)-Q_o(n)))*(hf_F(n)-
hf_F(n-1))/2;
end
a_steady=1;
while a_steady <= 500
        if hf_F(a_steady) > 0.01*h;
            a_steady=a_steady+1;
        else
            break
        end
end
        disp('Free response (Trapezoidal method),')
        disp(' 1% settling time in [s] = '),disp(t_F(a_steady))
        disp(' 1% steady-state liquid height in [m] ='),disp(hf_F(a_steady))
        disp(' 1% steady-state outflow rate Qo in [L/s] ='),disp(Q_oL(a_steady))

% Free Response Euler method
t_max=342;
delta_t=(t_max-0)/(ite-1);
t_E(1)=0;
h_E(1)=2;
Q_oE(1)=B*sqrt(h_E(1));
Q_oLE(1)=Q_o(1)*1000;
```

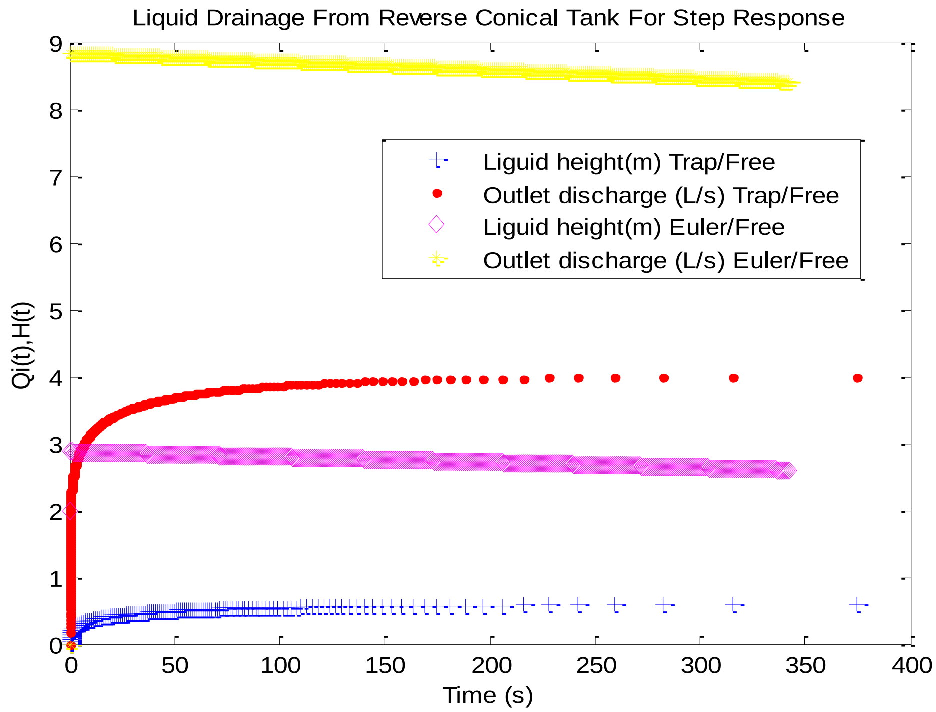
```
Q_iE(1)=0;
C_E(1)=pi*(R_m/H_m)^2*h_E(1)^2;

for n=2:ite;
    t_E(n)=0+(n-1)*delta_t;
    h_E(n)=h_E(n-1)+(Q_iE(n-1)-Q_oE(n-1))*delta_t/C_E(n-1);
    C_E(n)=pi*(R_m*h_E(n)/H_m)^2;
    Q_iE(n)=0;
    Q_iEL(n)=Q_iE(n)*1000;
    Q_oE(n)=B*sqrt(h_E(n));
    Q_oEL(n)=Q_oE(n)*1000;
end;
% Steady-state results (Euler method)
a_steady=1;
while a_steady <= 500
        if h_E(a_steady) > 0.01*h;
            a_steady=a_steady+1;
        else
            break
        end
end
        disp('Free response (Euler method),')
        disp(' 1% settling time in [s] = '),disp(t_E(a_steady))
        disp(' 1% steady-state liquid height in [m] ='),disp(h_E(a_steady))
        disp(' 1% steady-state outflow rate Qo in [L/s] ='),disp(Q_oEL(a_steady))

    plot(t_F,hf_F,'b+',t_F,Q_oL,'r.',t_E,h_E,'md',t_E,Q_oEL,'y*')
    legend('Liguid height(m) Trap/Free','Outlet discharge (L/s) Trap/Free','Liguid
height(m) Euler/Free','Outlet discharge (L/s) Euler/Free')
    title('Liquid Drainage From Reverse Conical Tank For free Response')
    xlabel('Time (s)')
    ylabel('Qi(t),H(t)')

% STEP RESPONSE ANALYSIS FOR INVERSE CONE
clear all
clc
% Intial Inputs:
C_d=0.6;    %Orifice draf cofficient
R_m=1;      %Radius of base of cone,m
H_m=2;      %Total height of cone,m
g=9.81;     %Acceleration due to gravity,m
d_o=0.05;   %Diameter of outlet,m


%Calculation
A_o=pi*((d_o)^2)/4; %cross sectional area of outlet, m^2

% Step Response Trapozoidal Method
h=0;                        %Initial height of liquid level,m
ite=501;                    %Number of steps
A=pi*(R_m/H_m)^2;           %Coffecient for fluid capacitance
B=C_d*A_o*sqrt(2*g);        %Coeffecient for outflow, m^(3/2)/s
Qi_s=4/1000;                %Inlet Discharge. m^3/s
h_max=(Qi_s/B)^2;   %Maximum Height of Fluid,m
h_step=(h_max-h)/(ite-1);    %Step size of height,m

for n=1:ite;
    hf_F(n)=h+(n-1)*h_step;
```

```
    Q_o(n)=B*sqrt(hf_F(n));
    Q_oL(n)=Q_o(n)*1000;
    Q_iL(n)=4;
    Q_i(n)=Q_iL(n)/1000;
    C_F(n)=pi*(R_m/H_m)^2*hf_F(n)^2;
end;

t_F(1)=0;
for n=2:ite
    t_F(n)=t_F(n-1)+(C_F(n-1)/(Q_i(n-1)-Q_o(n-1))+C_F(n)/(Q_i(n)-Q_o(n)))*(hf_F(n)-
hf_F(n-1))/2;
end
a_steady=1;
while a_steady <= 500
        if hf_F(a_steady) < 0.99*h_max
            a_steady=a_steady+1;
        else
            break
        end
end
        disp('Step response (Trapezoidal method),')
        disp(' 99% settling time in [s] = '),disp(t_F(a_steady))
        disp(' 99% steady-state liquid height in [m] ='),disp(hf_F(a_steady))
        disp(' 99% steady-state outflow rate Qo in [L/s] ='),disp(Q_oL(a_steady))

% Step Response Euler method
t_max=342;
delta_t=(t_max-0)/(ite-1);
t_E(1)=0;
h_E(1)=2;
Q_oE(1)=B*sqrt(h_E(1));
Q_oLE(1)=Q_o(1)*1000;
Q_iE(1)=4;
Q_iLE(1)=Q_i(1)/1000;
C_E(1)=pi*(R_m/H_m)^2*h_E(1)^2;

for n=2:ite;
    t_E(n)=0+(n-1)*delta_t;
    h_E(n)=h_E(n-1)+(Q_iE(n-1)-Q_oE(n-1))*delta_t/C_E(n-1);
    C_E(n)=pi*(R_m*h_E(n)/H_m)^2;
    Q_iEL(n)=4;
    Q_iE(n)=Q_iEL(n)/1000;
    Q_oE(n)=B*sqrt(h_E(n));
    Q_oEL(n)=Q_oE(n)*1000;
end;
% Steady-state results (Euler method)
a_steady=1;
while a_steady <= 500
        if h_E(a_steady) < 0.99*h_max;
            a_steady=a_steady+1;
        else
            break
        end
end
        disp('Step response (Euler method),')
        disp(' 99% settling time in [s] = '),disp(t_E(a_steady))
        disp(' 99% steady-state liquid height in [m] ='),disp(h_E(a_steady))
        disp(' 99% steady-state outflow rate Qo in [L/s] ='),disp(Q_oEL(a_steady))
```

```
    plot(t_F,hf_F,'b+',t_F,Q_oL,'r.',t_E,h_E,'md',t_E,Q_oEL,'y*')
    legend('Liguid height(m) Trap/Free','Outlet discharge (L/s) Trap/Free','Liguid
height(m) Euler/Free','Outlet discharge (L/s) Euler/Free')
    title('Liquid Drainage From Reverse Conical Tank For Step Response')
    xlabel('Time (s)')
    ylabel('Qi(t),H(t)')
```

# 5.0 Plots:



# 6.0 Analytical solution in Maple:

$$X := \pi \cdot \left(\frac{1}{2}\right)^2$$

$$\frac{1}{4}\pi$$

$$B := \frac{0.6 \cdot \pi \cdot .05^2}{4} \cdot \mathrm{sqrt}(2 \cdot 9.81)$$

$$0.001661042594\pi$$

$$H := 2$$

$$2$$

$$h := 0.01 \cdot H$$

$$0.02$$

$$Q := 0$$

$$0$$

$$t := \frac{2}{5}\,\frac{X\left(H^{5/2} - h^{5/2}\right)}{B} + \frac{1}{2}\,\frac{X\,Q\left(H^2 - h^2\right)}{B^2} + \frac{2}{3}\,\frac{X\,Q^2\left(H^{3/2} - h^{3/2}\right)}{B^3} + \frac{X\,Q^3\,(H - h)}{B^4} + \frac{2\,X\,Q^4\left(H^{0.5} - h^{0.5}\right)}{B^5} + \frac{2\,X\,Q^5 \ln\left(\frac{Q - B\,H^{0.5}}{Q - B\,h^{0.5}}\right)}{B^6}$$

$$240.8126086\sqrt{2} - 0.00340560456$$

$$evalf(\%)$$

$$340.557051$$

# 7.0 Results:

Free Response $H_{initial}$=2.0 m

| | $H_{initial}(1\%)$,m | $t_s(1\%)$,s |
|---|---|---|
| **Trapezoidal** | 0.02 | 340.56 |
| **Euler** | 0.07 | 342. |
| **Analytical** | 0.02 | 340.557 |

Step Response $Q_{inlet}$=4 [L/s]

| | $H_{ss}(99\%)$,m | $t_s(99\%)$,s | $Q_0(99\%), \frac{L}{s}$ |
|---|---|---|---|
| **Trapezoidal** | 0.58169 | 242.22 | 3.979 |
| **Euler** | 2/error | 0/error | 0/error |

Current Folder: C:\Users\pradip\Desktop

Command Window

```
Free response (Trapezoidal method),
 1% settling time in [s] =
      340.56

 1% steady-state liquid height in [m] =
        0.02

 1% steady-state outflow rate Qo in [L/s] =
     0.73798

Free response (Euler method),
 1% settling time in [s] =
   342

 1% steady-state liquid height in [m] =
    0.070726

 1% steady-state outflow rate Qo in [L/s] =
      1.3878

fx >>
```

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Command Window

```
Step response (Trapezoidal method),
 99% settling time in [s] =
      242.22

 99% steady-state liquid height in [m] =
     0.58169

 99% steady-state outflow rate Qo in [L/s] =
      3.9799

Step response (Euler method),
 99% settling time in [s] =
    0

 99% steady-state liquid height in [m] =
    2

 99% steady-state outflow rate Qo in [L/s] =
    0
```